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Multiple Category Memberships in Markets: An Integrative Theory and Two Empirical Tests

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This appendix contains the formal details of definitions, postulates, and theorems along with proof traces.

Key Functions

$\widetilde{\alpha}(l,x,y,t)$	intrinsic appeal of producer *x*'s offering in category *l* to audience member *y* at time *t*.
$\alpha(l,x,y,t)$	actual appeal to audience member *y* of producer *x*'s offering in category *l* at time *t*
$\mathrm{E}(\cdot)$	mathematical expectation
$\mathbf{l}_p(m,t)$	set of positively valued categories (within some unspecified super-category) in the market *m* at time *t*
$\mu_{i(l)}(x,y,t)$	grade of membership of producer *x* in the "meaning" (or intension) of the label *l* from audience member *y*'s perspective at time point *t*
$\nu_{i(l)}(y,t)$	grade of membership of audience member *y* in the consensus in an audience about the meaning (intension) of the label *l* at time *t*.

Key Predicates

$\mathrm{CAT}(l,m,t)$	the audience in the market *m* has reached a high level of consensus about the meaning of the category with the label *l* at time *t*.
$\mathrm{TY}(\mathbf{l},y,t)$	the agent *y* is a typical member of the audience for the superset of categories **l** at time *t*.

To develop our theorems, we use a nonmonotonic logic developed by Pólos and Hannan (2002, 2004). (In logic, nomonotonicity means that adding premises to an argument might kill implications of the unaugmented argument.) This logic is designed for testing the validity of inferences from arguments that build on *rules-with-exceptions*. The logic uses a quantifier, denoted by $\mathfrak{N}$, that parallels the standard universal quantifier of first-order logic ($\forall$). It tells what is normally the case. Because more specific information can override the implications of a set of rules-with-exceptions, the implications (lemmas and theorems) have a different status than the premises. This difference is marked syntactically in the logic by the fact that it uses a different quantifier, $\mathfrak{P}$, to tell what follows as presumedly the case at the present stage of a theory. Finally, the logic marks syntactically the difference between substantive premises, quantified by $\mathfrak{N}$, and auxiliary assumptions by using the quantifier $\mathfrak{A}$ for the latter.

Definition 1 (Positively-valued category)**.** *A positively-valued category is one for which the typical members of the audience find intrinsically more appealing the offerings of more clear-cut members of the category, those with higher GoM in the meaning of the category,* $\mu_{i(l)}$.

$$\text{PCAT}(l,m,t) \leftrightarrow \text{CAT}(l,m,t) \wedge \mathfrak{N} t,x,x',y\,[\,\text{TY}(\mathbf{l},y,t) \wedge (\mu_{i(l)}(x,y,t) > \mu_{i(l)}(x',y',t)) \\ \rightarrow \text{E}(\widetilde{\alpha}(l,x,y,t)) > \text{E}(\widetilde{\alpha}(l,x',y,t))].$$

[Read: l is a positively-valued category in the market m at time t if and only if l is a category in the market m at time t and it is normally the case, for all time points, pairs of producers, and audience members, that if the audience member is typical of the audience for the market at that time point and the audience member assigns a higher GoM in the category to one producer than another, then the expected actual appeal to that audience member is higher for producer with the higher GoM.] Henceforth we concentrate on positively-valued categories in a market; and we denote this set by $\mathbf{l}_p(m,t)$.

Definition 2 (Category engagement)**.**

A. *The level of category-engagement function,* $en(l,x,t)$*, maps four-tuples of categories, producers, and time points to the nonnegative real line. This function gives the level of the engagement of producer* x *in category* l *at time* t.

B. *The grade-of-membership function for engagement in categories,* $\varepsilon(l,x,t)$*, is the share of producer* x*'s engagement in category* l *at time* t.

$$\varepsilon(l,x,t) = \frac{en(l,x,t)}{En(x,t)}, \quad \text{where } En(x,t) = \textstyle\sum_{l \in \mathbf{l}_p(m,t)} en(l,x,t).$$

Postulate 1 (For positively-valued categories, expected actual appeal increases with intrinsic appeal and engagement)**.**

A. *The expected actual appeal of an offering in a category to an audience member normally equals zero if its intrinsic appeal is zero:* $\widetilde{\alpha}(l,x,y,t) = 0$ *or the producer's engagement in the category is zero:* $\varepsilon(l,x,t) = 0$.

B. *The expected actual appeal of an offer in a positively-valued category to an audience member normally increases with its intrinsic appeal (as long as engagement is nonzero).*

C. *The expected actual appeal of an offer in a positively-valued category normally increases with the producer's engagement in a category (as long as its intrinsic appeal exceeds zero).*

Postulate 2 (Principles of allocation)**.**

A. *The expected sum of total category memberships to a typical audience member is the same for all producers in a market.*

$$\mathfrak{N}m\,\exists\,\mathscr{M}_m \forall t,x,x',y\,[\,\mathrm{TY}(\mathbf{l}_p,y,t) \rightarrow (\mathrm{E}(M(x,y,t)) = \mathscr{M}_m = \mathrm{E}(M(x',y,t)))],$$

where $M(x,y,t) = \sum_{l\in\mathbf{l}_p(m,t)} \mu_{i(l)}(x,y,t)$.

B. *The expected level of total category engagement is the same for all producers in a market.*

$$\mathfrak{N}m\,\exists\,\mathscr{E}_m \forall t,x,x'\,[\,\mathrm{E}(En(x,t)) = \mathscr{E}_m = \mathrm{E}(En(x',t))\,].$$

[Read for part A: it is normally the case for all markets that there exists a constant $\mathscr{M}_m$ such that for all time points, pairs of producers, and audience members that if the audience member is typical then the expectation of the sum of the GoMs in the market categories that it assigns to the producers equals the constant.]

Auxiliary assumption 1. *The maximal sum of a producer's GoMs in the categories in market's superset of categories equals 1.*

$$\mathfrak{A}m\,[\mathscr{M}_m \leq 1].$$

[Read: as an auxiliary assumption, the constant $\mathscr{M}_m$ cannot exceed unity for any market.[1]]

Definition 3 (Niches in categories)**.**

A. *A producer's category-membership niche to an audience member is a fuzzy set whose GoM function is its GoM in each category from perspective of the agent.*

$$\mu(x,y,t) = \{l, \mu_{i(l)}(x,y,t)\}, \quad l \in \mathbf{l}_p(m,t).$$

B. *A producer's category-engagement niche is a fuzzy set whose GoM function in a category is the proportion of its engagement that it devotes to the category.*

$$\varepsilon(x,t) = \{l, \varepsilon(l,x,t)\}, \quad l \in \mathbf{l}_p(m,t).$$

Definition 4 (Niche width)**.**

A. *Width of a category-membership niche:*

$$wd\,\mu(x,y,t) = 1 - \textstyle\sum_{l \in \mathbf{l}_p(m,t)} \widetilde{\mu}^2_{i(l)}(x,y,t),$$

where $\widetilde{\mu}_{i(l)}(x,y,t) = \mu_{i(l)}(x,y,t)/\mathscr{M}_m$.

B. *Width of a category-engagement niche:*

$$wd\,\varepsilon(x,t) = 1 - \textstyle\sum_{l \in \mathbf{l}_p(m,t)} \varepsilon^2(l,x,t).$$

Theorem 1.

Let l be a positively-valued category in the market m at time t: $l \in \mathbf{l}_p(m,t)$.

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[1] Auxiliary assumptions are premises that theorists introduce into arguments to link causal stories and desired theorems. They often take the form of some simplifying assumptions, descriptions of constraints that make mathematical modeling possible. Such assumptions are not persistent in an evolving theory, because they are made for special purposes but are not claimed to be causal insights.

A. *A category-membership specialist has a higher average GoM in its focal category than any category-generalist.*

$$\mathfrak{P}\, l,m,t,x,x',y\,[\,(\mu_{i(l)}(x,t)=1)\wedge(wd\,\mu(x,y,t)<wd\,\mu(x',y,t))\rightarrow(\mu_{i(l)}(x,y,t)>\mu_{i(l)}(x',y,t))].$$

B. *A category-engagement specialist has a higher level of engagement in its focal category than any engagement-generalist.*

$$\mathfrak{P}\, l,m,t,x,x'\,[\,(\varepsilon(l,x,t)=1)\wedge(wd\,\varepsilon(x,t)<wd\,\varepsilon(x',t))\rightarrow(\varepsilon(l,x,t)>\varepsilon(l,x',t))].$$

Proof. In the nonmonotonic logic used (Pólos and Hannan 2004), a proof involves collecting the "rule chains" (links of postulates, auxiliary assumptions, and definitions) that connect the antecedent and consequent in the theorem. A theorem is proven if a rule chain can be found that makes the claimed connection and there is no rule chain leading to the opposite inference[2]that is more specific or incomparably specific. In the case of these theorems, all of the rule chains point in the same "direction." Therefore, we sketch the minimal rule chain that constitutes a proof.

Part A. By Definition 4A, $wd\,\mu(x,y,t) < wd\,\mu(x',y,t)$ and $\mu_{i(l)}(x,y,t) = 1$ imply that there is some category $l' \neq l$ for which x' has nonzero membership from y's perspective (otherwise the niche widths could not differ). The principle of allocation, Postulate 2A, imposes the restriction that the sum of category memberships is the same for all producers in the market. Then the consequent in the theorem follows immediately.

The proof of Part B parallels that of Part A; but it uses Definition 4B and Postulate 2B. □

Theorem 2 (Specialist appeal advantage)**.**

Let l be a positively-valued category in the market m at time t and y be a typical member of the audience.

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[2]The theorems in this paper claim a positive monotonic relationship between function: the larger the ϕ, the larger the ψ. The opposing claim would hold either that there is no relation between ψ and ψ or that there is a *negative* monotonic relationship between them.

A. *The expected actual appeal of the offering of a category-membership specialist in its focal category exceeds that any generalist (for typical members of the audience), as long as the specialist engages its focal category.*

$$\mathfrak{P}\, l,m,t,x,x',y\,[\,(\mu_{i(l)}(x,y,t)=1)\wedge(\varepsilon(l,x,t)>0)\wedge(wd\,\mu(x,t)<wd\,\mu(x',t)) \rightarrow \mathrm{E}(\alpha(l,x,y,t))>\mathrm{E}(\alpha(l,x',y,t))];$$

B. *the expected actual appeal of the offering of a category-engagement specialist in its focal category exceeds that any generalist (for typical members of the audience), as long as the offering has nonzero intrinsic appeal in the category in which the specialist focuses its engagement.*

$$\mathfrak{P}\, l,m,t,x,x',y\,[\,(\varepsilon(l,x,t)=1)\wedge(\mu_{i(l)}(x,y,t)>0)\wedge(wd\,\varepsilon(x,t)<wd\,\varepsilon(x',t)) \rightarrow (\mathrm{E}(\alpha(l,x,y,t))>\mathrm{E}(\alpha(l,x',y,t)))].$$

Proof. Part A. The only rule chains that connect the antecedent and consequent using the available premises yield the theorem. The minimal rule chain builds on the rule chain supporting Theorem 1A (a specialist has higher GoM in its focal category than any category generalist) the definition of a positively-valued category, Definition 1 (expected actual appeal of a producer's offering increases with a producer's GoM in a category for typical audience members), and Postulate 1B (expected actual appeal of a producer's offering increases with intrinsic appeal, given nonzero engagement).

The proof of Part B parallels that of Part A; but it uses Theorem 1B and Postulate 1C. □

Definition 5 (Diverse market)**.** *A market is diverse if it is normally the case, for each (positively valued) category and each typical audience member, that at least one producer with maximal grade of membership in the category also engages the category.*

$$\textsc{Div}(m,t)\leftrightarrow\forall\, l,y\,[\,(l\in\mathbf{l}_p(m,t))\wedge\textsc{ty}(\mathbf{l}_p,y,t)\rightarrow\exists\, x\,[\,(\mu_{i(l)}(x,y,t)=1)\wedge(\varepsilon(l,x,t)>0)]].$$

Theorem 3 (Generalists lack high appeal in diverse markets)**.**

Let l be a positively-valued category in the market m at time t and y be a typical member of the audience.

A. *In a diverse market, generalists in category membership have lower expected actual appeal than at least one producer in every category (to typical audience members and positively-valued categories).*

$$\mathfrak{P}\, l,m,t,x,y\, [\textsc{div}(m,t) \wedge (wd\,\mu(x,y,t) > 0) \rightarrow \exists x' \,[\mathrm{E}(\alpha(l,x,y,t)) < \mathrm{E}(\alpha(l,x',y,t))]].$$

B. *In a diverse market, generalists in category engagement have lower expected actual appeal than at least one producer in every category (to typical audience members and positively-valued categories).*

$$\mathfrak{P}\, l,m,t,x,y\, [\textsc{div}(m,t) \wedge (wd\,\varepsilon(x,t) > 0) \rightarrow \exists x' \,[\mathrm{E}(\alpha(l,x,y,t)) < \mathrm{E}(\alpha(l,x',y,t))]].$$

Proof. Part A. Again the available premises yield only rule chains that connect the antecedent and consequent as stated. The definition of diversity, Definition 5, guarantees that there is a category-specialist in each category from the perspective of each typical member of the audience and that this specialist engages the category. It also restricts the scope to typical audience members. Given that the focal producer is a generalist, in the sense that the width of its category-membership niche exceeds zero, then (the rule chain warranting) Theorem 1A applies. It states that any category specialist has higher GoM in the category than the focal generalist. The definition of a positively valued category, Definition 1, states that expected actual appeal of a producer's offering increases with a producer's GoM in a category for typical audience members. Finally, Postulate 2A tells that, given nonzero engagement, the expected actual appeal of an offering with higher intrinsic appeal exceeds that of an offering with less intrinsic appeal.

The proof of Part B exactly parallels that of Part A, except that it uses the rule chains warranting Theorem 1B. □

Definition 6 (Fitness in a category)**.** *A producer's relative fitness in a category is its share of the total appeal of its offerings to the typical members of the audience as contrasted with the appeals* *of all of the offerings in the category.*

$$\phi(l,x,t) = \frac{Ap(l,x,t)}{\sum_{x'} Ap(l,x',t)},$$

where $Ap(l,x,t)$ denotes x's total appeal in category l at time t to the typical members of the audience for the category, i.e.,

$$Ap(l,x,t) = \textstyle\sum_{y|\mathrm{TY}(l,y,t)} ap(l,x,y,t).$$

Postulate 3. *A producer's expected fitness in a positively-valued category normally increases monotonically with the total appeal of its offerings in that category.*

$$\mathfrak{N}\, l,t,x,x'\,[\,(l \in \mathbf{l}_p(m,t)) \wedge (Ap(l,x,t) > Ap(l,x',t)) \rightarrow (\mathrm{E}(\phi(l,x,t)) > \mathrm{E}(\phi(l,x',t)))\,].$$

Theorem 4 (Generalist fitness in diverse markets)**.**

Let l be a positively-valued category in the market m at time t and y be a typical member of the audience.

A. *In a diverse market, generalists in category membership have lower expected fitness in all positively-valued categories than at least one producer (to typical audience members).*

$$\mathfrak{P}\, l,m,t,x\,[\,\mathrm{DIV}(m,t) \wedge (wd\,\mu(x,y,t) > 0) \rightarrow \exists x'\,[\mathrm{E}(\phi(l,x,t)) < \mathrm{E}(\phi(l,x',t))]].$$

B. *In a diverse market, generalists in category engagement have lower expected fitness in all positively-valued categories than at least one producer (to typical audience members).*

$$\mathfrak{P}\, l,m,t,x\,[\,\mathrm{DIV}(m,t) \wedge (wd\,\varepsilon(x,t) > 0) \rightarrow \exists x'\,[\mathrm{E}(\phi(l,x,t)) < \mathrm{E}(\phi(l,x',t))]].$$

Proof. The minimal rule chain that connects the antecedents and consequents uses (the rule chains that warrant) Theorem 3 along with Definition 6 and Postulate 3 (and summation over the typical members of the market). □

References

Pólos, László and Michael T. Hannan. 2002. Reasoning with partial knowledge. *Sociological Methodology* 32:133–81.

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